

Classifying Points for Sweeping Solids*

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Abstract

Many diverse engineering problems can be modeled with solid sweeping in a conceptually simple and intuitive way, and sweeps are considered to be one of the basic representation schemes in solid modeling. However, many properties of sweeps as well as their “informational completeness” are not well understood, which is the primary reason why computational support for solid sweeping remains scarce.

We propose a generic point membership classification (PMC) for sweeping solids of arbitrary complexity moving according to one parameter *affine* motions. The only restrictive assumption that we make in this paper is that the initial and final configurations of the moving object do not intersect during the sweep. Our PMC test is defined in terms of inverted trajectory tests against the *original* geometric representation of the generator object, which implies that this test can be implemented in any geometric representation that supports curve-solid intersections. Importantly, this PMC test provides complete geometric information about the set swept by 3-dimensional objects in general motions. At the same time, it establishes the foundations for developing a new generation of computational tools for sweep boundary evaluation and trimming, as well as a number of practical applications such as shape synthesis, contact analysis and path planning.

1 Introduction

A set of points X moving according to a prescribed motion M in an d -dimensional Euclidean space \mathbb{E}^d occupies a subset of \mathbb{E}^d widely known as the set swept by X during M , and denoted by $sweep(X, M)$. Sweeping a set of points is one of the fundamental representations in solid and geometric modeling [1] with a diverse set of applications, such as solid modeling, robotics, NC machining and path planning, collision detection, computer assisted surgery, and ergonomics.

The mathematical foundations of sweeps relying on the envelope and singularity theory reveal the high level of complexity of general sweeping of sets of points in space [2]. Despite the fact that these foundations seem to be understood fairly well, the validity and computational properties of sweeps are not. Hence, it should be no surprise that commercial solid modeling codes have only limited capabilities, if any, for sweeping solids.

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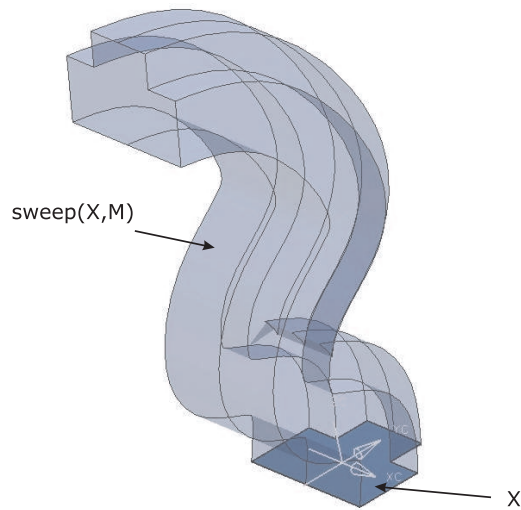


Figure 1: Sweeping a 2-dimensional cross-section is a special case of a more general sweep operation.

Many approaches to compute the boundary of the sweep have been proposed in the literature that make assumptions on and exploit the geometric and motion representations of the moving object, and are therefore restricting the class of sweeps that they can handle. For example, many published algorithms

- o restrict the type of moving solid X , for example to a convex polyhedron, or a planar cross-section moving in 2D or 3D [3, 4, 5] as illustrated in Figure 1;
- o limit the motion to pure translations, rotations, or screw motions, and do not allow motions M that produce self intersections of the sweep [3, 6, 7];
- o approximate the set swept by the moving object as a discrete union of instances of the object sampled along the motion [8];
- o use rendering methods to compute the image of the sweep without computing the complete representation of the sweep [9];
- o formulate the sweep of implicitly defined shapes [10, 11, 12, 13];

The computational difficulties of sweeps are apparent if one observes that the sweep boundary evaluation should produce points that are solutions to systems of nonlinear differential equations [14, 15]. Since the envelopes cannot be generated in closed form for any case of reasonable complexity, a number of algorithms have been proposed to approximate the envelope surfaces for implicit and parametric surfaces [16]. Differential conditions for generating points on the boundary of the sweep have been presented in [17, 11, 12], while techniques for generating envelopes of “specialized” curves and surfaces, such as envelopes of planes, and quadrics have been presented in [14]. A recent review of the underlying mathematical foundations is given in [18].

The ability to classify a point against a solid as “in”, “on” or “out” of the solid, is a sufficient test for geometric completeness of a representation, in the sense that any geometric property can, in principle, be computed [1, 19]. Such a test is commonly known as Point Membership Classification (PMC) test. In other words, defining a PMC test for sweeps would allow one to compute, in principle, all the points swept by a moving set, including those that are “on” the boundary of the sweep, or envelope points that are not part of the sweep boundary. As is the case with other solid representations and operations with solid objects, the availability of a PMC test for sweeps could be used to compute the boundary of the sweep exactly, within the machine precision, or approximately.

Efficient algorithms for approximating the boundary of the volume swept by polyhedral objects have been presented, for example, in [5]. This work does not define a point membership test for polyhedral objects. Their algorithm performs the boundary trimming by taking advantage of the simplified geometry of the generator to compute a signed distance field relative to the ruled and developable surfaces that are generated by the moving triangles, but cannot be applied to objects bounded by curved surfaces.

More broadly, PMC tests for general solid sweeps and motions have not yet been published in the literature. Instead, PMC tests have been suggested for restricted classes of shapes in [9]. Furthermore, it has been argued in [20] that a PMC test for general sweeps can be formulated based on the duality between the sweep and unsweep operations, where it was stated that a stationary point x belongs to the sweep of moving X if and only if the inverted trajectory of x penetrates X at its initial position.

A recent paper [7] proposes the boundary evaluation of the set swept by a moving solid whose movement is restricted to a screw motion. The authors take advantage of the fact that the so called “grazing points¹” are fixed on the moving object throughout a screw motion. They use an observation similar to the one described in [20], and compute inverted trajectories of representative points of the candidate boundary faces to decide which of these faces are part of the boundary of the sweep. However, their work is restricted to screw motions, they do not give details about how they process the information obtained from the trajectory intersection, and do not go beyond testing carefully chosen candidate boundary points.

The research described in this paper goes significantly beyond the work described in [7] in several different ways. First, we do not restrict the motions to be screw motions, but we consider more general affine motions that do not have to be rigid. Second, we present the first point membership classification test that is valid not only for any and all sweep boundary points (that are of interest in most sweep related research), but also for all other points of the space in which the solid moves. In other words, we provide a finer decomposition of space than the usual PMC tests for geometric objects, and show that our finer decomposition can provide solutions to a number of open engineering problems such as shape synthesis of higher pairs, contact analysis and path planning. Third, we show that our classification is complete in the sense that we capture all possible interactions between inverted trajectories and the object at its initial configuration.

1.1 Goals and Outline

In this paper, we propose a generic Point Membership Classification test for sweeps generated by 3-dimensional solids moving according to one-parameter affine motions. The only constraint that

¹See [17, 11] for a detailed discussion of the grazing points.

we place on the motion in this paper is that the initial and final configurations of the moving set do not intersect. We show that this test can be implemented for any geometric representation that supports curve-solid intersections, and we describe our prototype implementation for solids given in a boundary representation (B-rep) within a commercial geometric kernel. A short version of this paper appeared in [21]. Here we extend that work and introduce a simpler classification of points relative to the set swept by a moving generator, which is based on a new approach to count the points of intersection and tangency between inverted trajectories and the generator. We formulate our approach in section 2, and discuss the completeness of our classification in section 3. We illustrate several different 3-dimensional examples in section 4, and summarize in section 5 the contributions of this paper as well as the importance of the PMC test for general sweeps.

2 The Point Membership Test

The standard PMC test of a point P against a set X is defined as [19]

$$PMC(P, X) = \begin{cases} in, & \text{if } P \in \mathbf{i}X \\ on, & \text{if } P \in \partial X \\ out, & \text{if } P \in X^c \end{cases} \quad (1)$$

where ∂X is the boundary of X , and X^c is the complement of X . Not only that the ability to perform a PMC test for a given representation ensures that the representation is unambiguous, but any geometrically complete representation has to support, by definition, such a PMC test.

Sweeping an arbitrary set of points S along a motion M in a d -dimensional Euclidean space \mathbb{E}^d is expressed as an infinite union operation:

$$sweep(S, M) = \bigcup_{q \in M} S^q, \quad (2)$$

where S^q denotes the set S positioned according to a configuration q of motion $M(t)$, and $t \in [0, 1]$ is the motion parameter within a normalized interval². Sweeps are widely used as an effective design tool for creating highly complex three-dimensional shapes, but performing boundary evaluation for sweeps remains difficult. In fact, general sweeps of 3D solids are not implemented today in commercial CAD packages due to well-known and well-documented computational issues.

In this paper we focus on the case where set S is a d -dimensional solid (as defined in [1]) moving in a d -dimensional Euclidean space \mathbb{E}^d . We assume that motion M is a one parameter family of affine transformations $M(t)$, with $t \in [0, 1]$. All points of S will move through space describing a trajectory T determined by transformation $M(t)$ in an absolute coordinate system, and the trajectory of a point Q of S can be written as

$$T_Q = \{Q^q \mid Q \in S, q \in M\}. \quad (3)$$

Clearly, curve T_Q contains all points of the space that will be occupied by point Q at *some* parameter value during the motion. However, when observed from a moving coordinate system rigidly attached

²Observe that normalized parameter intervals are merely used for computational convenience, and do not restrict the problem under consideration.

to the moving set S , the same point Q will appear to describe a different trajectory denoted by \hat{T}_Q . This *inverted trajectory* can be written as

$$\hat{T}_Q = \{Q^p \mid Q \in S, p \in \hat{M}\} \quad (4)$$

where \hat{M} is the *inverted motion*, i.e., $\hat{M}(t)$ is the inverse of $M(t)$ for every value of t [20]. In fact, one can show that the inverted trajectory \hat{T}_Q contains **all** points $y \in \mathbb{E}^d$ that will pass through the given point Q when moved according to motion M [22].

In other words, if the inverted trajectory of a given point $P \in \mathbb{E}^d$ intersects S in its original configuration, or

$$\hat{T}_P \cap S^0 \neq \emptyset \quad (5)$$

then *all* the corresponding intersection points will pass through P at various parameter values and therefore P is either *in* or *on* the set $\text{sweep}(S, M)$. By contrast, if

$$\hat{T}_P \cap S^0 = \emptyset \quad (6)$$

then there are no points of S that will pass through P during M and, therefore, point P does not belong to the sweep.

In what follows, we show that the differentiation between the *in* and the *on* points of $\text{sweep}(S, M)$ can be achieved by counting how many points of set $\hat{T}_P \cap S^0$ from equation (5) are *intersection* or *tangency* points with the boundary ∂S of S . The way we count these intersection and tangency points, as well as the reasons behind this particular choice, are discussed in section 2.1. Consequently, we denote the number of intersection points between \hat{T}_P and S^0 by i_p , and the number of corresponding tangency points by t_p .

There are multiple ways in which a given continuous curve can intersect a known solid. Figure 2 illustrates all the “primitive” types of intersections in the sense that every other curve-solid intersection can be expressed as a combination of these cases. Generally speaking, a rigorous implementation of all these cases would require neighborhood computations, for example, when the intersection or tangent point lies on an edge or vertex of the solid. Alternatively, one can avoid the difficulties in representing point neighborhoods for such tests by testing points of the curve in the neighborhood of the intersection or tangency point relative to the solid itself (see also [23]).

Consider the example shown in Figure 3 in which an object S sweeps the shaded set of points, which is shown here in 2D for simplicity. The boundary of S will generate envelopes that are curves in 2D or surfaces in 3D. A subset of these envelopes will be part of the final boundary of $\text{sweep}(S, M)$, while the remaining envelopes will lie inside the set swept by S . We call the latter the *internal* envelopes. In addition, the sweep boundary will contain portions of the boundary of S positioned at $t = 0$ and $t = 1$ (see for example [11]).

The point membership test formulated in this paper induces a complete disjoint decomposition of the space in which the object moves. Classifying a point with respect to this decomposition depends on all possible positions of a point P relative to S^0 and S^1 as summarized in table 1:

2.1 Counting the intersection and tangency points

Our PMC relies on determining the number of intersection and tangency points between the inverted trajectory \hat{T}_P of point P and the generator S^0 based on the duality discussed in [24]. For most points P , the intersection and tangency points of their inverted trajectories follow established topological

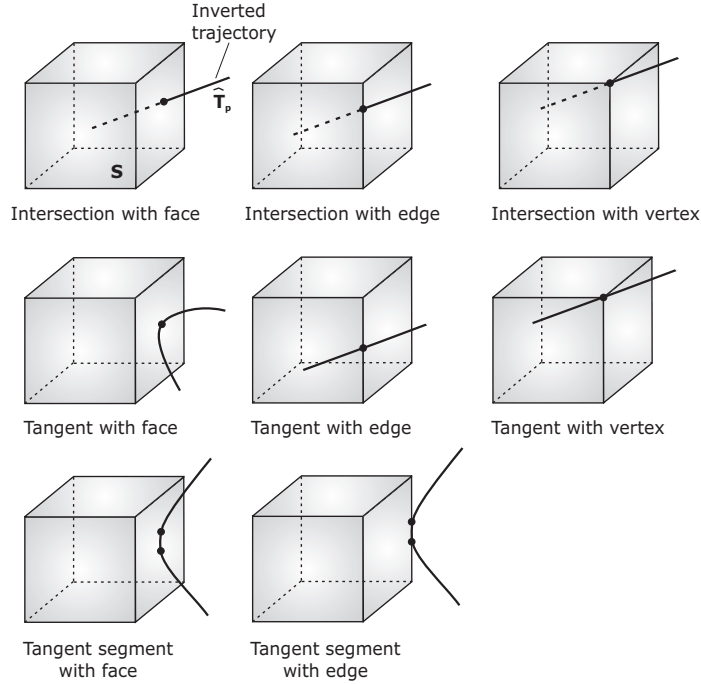


Figure 2: “Primitive” intersection and tangency cases between a curve (such as the inverted trajectory \hat{T}_P) and the boundary of a solid S . All other cases can be expressed as a combination of these cases.

conditions in terms of point neighborhoods [25, 26]. However, the inverted trajectories of points that are on the boundary of either S^0 or S^1 will “start” or “end” on S^0 , and there is more than one interpretation of the tangency or intersection properties of these points relative to S^0 . Importantly, such specific choices influence the complexity of not only the definitions of this PMC, but also of the algorithms that implement such a classification.

Consider the example shown in Figure 4(a), where a solid 2D shape is translating in the same plane³. The inverted trajectory of point P_0 intersects S^0 twice, and therefore $i_p = 2$, and $t_p = 0$. Furthermore, P_1 has $i_p = 8$, and $t_p = 1$, while point P_2 has $i_p = 2$, and $t_p = 4$. Similarly, one can in principle compute i_p and t_p for any other point in that space given the inverted trajectory \hat{T}_P

³We use 2D shapes and translations for illustrative purposes; this discussion applies to both 2 and 3 dimensional solids.

and	OUT S^0	ON S^0	IN S^0
OUT S^1	Case1	Case4	Case7
ON S^1	Case2	Case5	Case8
IN S^1	Case3	Case6	Case9

Table 1: Possible classification cases.

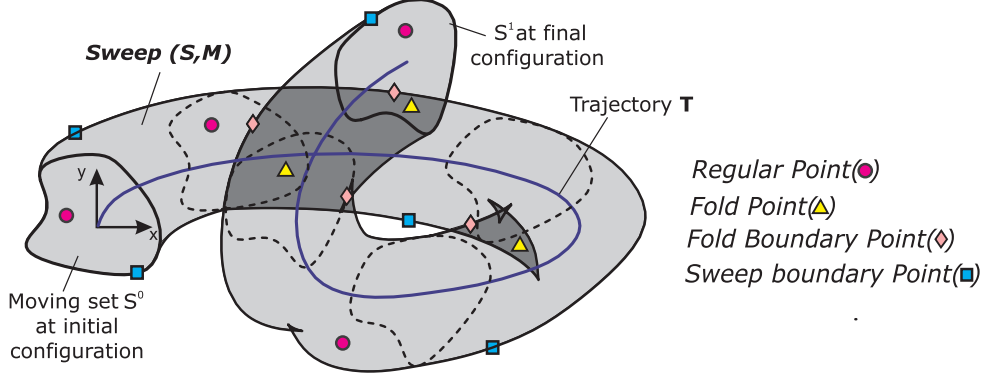


Figure 3: The sweep of a set S according to a rigid body motion consisting of a translation along trajectory T and rotation by $\pi/2$.

of P and the generator S^0 in its initial configuration. Thus, point P_4 would have $i_p = 3$, $t_p = 1$, while $i_p = 6$, $t_p = 1$ would correspond to P_3 . Note that both these points are on the boundary of S^1 (i.e., S at configuration $M(1)$), and therefore the other endpoint of their inverted trajectories will be, by definition, on the boundary of S^0 .

Since the inverted trajectory of P_4 does not “cross” the boundary of S^0 at $Q_4 = \hat{M}(1)P_4$, it is reasonable to ask whether Q_4 is really an intersection point or a tangency point of the inverted trajectory of P_4 . The tangent to the inverted trajectory at Q_4 is obviously not in the tangent plane to the boundary of S^0 at that point. Thus, based on this test, Q_4 is an intersection point. On the other hand, Q_4 can be classified as a tangency point if we examine the inverted trajectory of P_4 in the dual higher dimensional space (see [24] for a discussion of this duality) where this trajectory is a line parallel to the t axis as shown in Figure 4(b). Such a line L_4 that passes through P_4 will intersect the higher dimensional surface Σ two times, and has two tangencies with Σ ; one of these tangent points is where L_4 “touches” the boundary of S^1 . Thus, by using this information from the dual space, the inverted trajectory of P_4 would have $i_p = 2$ and $t_p = 2$, which is different than before. Furthermore, line L_3 passing through a point $P_3 \in \partial S^1$ would intersect Σ six times and would have one tangency point with Σ , which is consistent with the counting described above. A similar discussion applies to points that are on the boundary of S^0 .

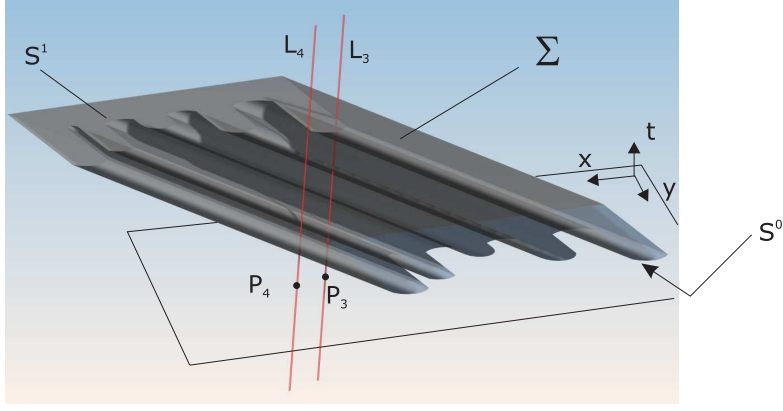
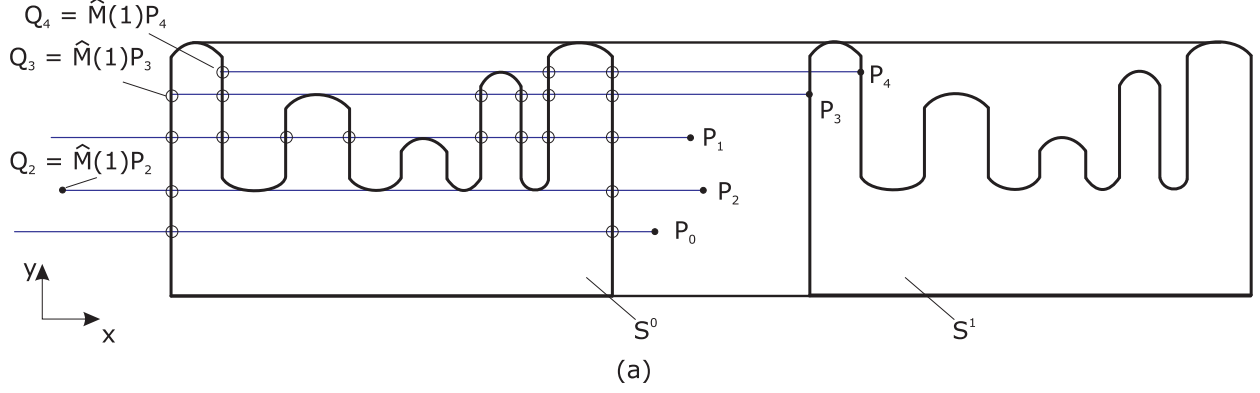
We now provide criteria for deciding whether the end points of the inverted trajectories \hat{T}_P of points P that are on the boundary of S^0 or S^1 are counted as intersection or tangency points. The proposed criteria are based on the duality between the d -dimensional Euclidean space of the motion and the $d + 1$ higher dimensional space in which Σ is generated [24]. We use the notion of an open ball of a point P :

Definition 1 An open ball of radius $\rho > 0$ centered at a point P of \mathbb{E}^d is the set

$$B(P, \rho) = \{x \in \mathbb{E}^d : \text{distance}(x, P) < \rho\}$$

where $\text{distance}(x, P)$ represents the Euclidean distance, and $d = 2, 3$. Consequently,

Definition 2 An open neighborhood $N(P, X)$ of a point P with respect to solid X is the intersection of an open ball of infinitesimal radius ρ centered at P with the interior iX of X .



(b)

Figure 4: Counting the number of intersections and tangencies between inverted trajectories and S^0 . Points that are on the boundary of S^0 and S^1 start or end on S^0 .

Denote by $Q = \hat{M}(1)(P)$ the second endpoint of the inverted trajectory \hat{T}_P (the first endpoint is P); by $t(\hat{T}_P, Q)$ the tangent to the inverted trajectory \hat{T}_P at point Q , and recall that if $P \in \partial S^1$, then $Q = \hat{M}(1)P$ will be on the boundary of S^0 (see points P_3 and P_4 in Figure 4(a)). Therefore,

Definition 3 *If point $P \in \partial S^1$ is a boundary point of S^1 , then point $Q = \hat{M}(1)P$ is an intersection point of \hat{T}_P iff*

$$N(Q, S^0) \cap \hat{T}_P \neq \emptyset.$$

Otherwise Q is a tangency point of \hat{T}_P .

Furthermore,

Definition 4 *Point $P \in \partial S^0$ that is a boundary point of S^0 is an intersection point of \hat{T}_P iff*

$$N(P, S^0) \cap \hat{T}_P \neq \emptyset.$$

Otherwise P is a tangency point of \hat{T}_P .

Based on these definitions, the inverted trajectory of point P_3 in Figure 4 will have $i_p = 6$ and $t_p = 1$, while that of point P_4 will have $i_p = 2$ and $t_p = 2$. We now define the interior points of the sweep (i.e., regular, fold, and fold boundary points), the sweep boundary points, and the exterior points of the sweep.

2.2 The interior points of sweep

The interior points of sweep can be grouped into three major categories⁴: (1) *regular* points, (2) *fold boundary* points that are on the internal envelopes or part of ∂S^0 or ∂S^1 ; and (3) *fold* points - which are the interior points bounded by the fold boundary points. We observe in Figure 3 that the regular points themselves can be classified further depending on their position relative to the original and final configuration of S . Since we assumed that our motion is such that $S^0 \cap S^1 = \emptyset$, cases 5, 6, 8, and 9 shown in table 1 will not occur. Thus,

Definition 5 *Given a solid d -dimensional set S and a motion M in \mathbb{E}^d such that $S^0 \cap S^1 = \emptyset$, a point $P \in \mathbb{E}^d$ is a **regular point** of the set sweep(S, M) iff one of the following conditions hold:*

Case 1: $P \notin S^0, P \notin S^1, i_p = 2$ and $t_p = 0$;

Case 2: $P \notin S^0, P \in \partial S^1, i_p = 2$ and $t_p = 0$;

Case 3: $P \notin S^0, P \in iS^1, i_p = 1$ and $t_p = 0$;

Case 4: $P \in \partial S^0, P \notin S^1, i_p = 2$ and $t_p = 0$;

Case 7: $P \in iS^0, P \notin S^1, i_p = 1$ and $t_p = 0$.

The second type of interior points are the (so-called) *fold points*, which can be found, for example, in the neighborhood of envelope singularities of the moving S (as illustrated in Figure 3). These fold points are closely related to the singularities in the catastrophe surface where the surface folds onto itself when projecting onto a lower dimensional space [27]. Intuitively, these points are the regions of space swept by subsets of the moving object more than once, and the existence of these fold regions is a necessary but not sufficient condition for the existence of critical points of the envelopes [24]. We define the fold points in terms of the *inverted trajectory-solid intersection*, and classify them into different categories depending on the position of the test point P with respect to S^0 and S^1 as shown in Figure 5(c-f). Let $k, j \in \mathbb{N}$ be two natural numbers. Then,

Definition 6 *Given a solid d -dimensional set S and a motion M in \mathbb{E}^d such that $S^0 \cap S^1 = \emptyset$, a point $P \in \mathbb{E}^d$ is a **fold point** of the set sweep(S, M) iff one of the following conditions hold:*

Case 1: $P \notin S^0, P \notin S^1, i_p = 2k$ and $t_p = 0$ for $k \geq 2$;

Case 2: $P \notin S^0, P \in \partial S^1, i_p = 2k$ and $t_p = 0$ for $k \geq 2$;

Case 3: $P \notin S^0, P \in iS^1, i_p = 2k - 1$ and $t_p = 0$ for $k \geq 2$;

Case 4: $P \in \partial S^0, P \notin S^1, i_p = 2k$ and $t_p = 0$ for $k \geq 2$;

⁴This specific choice of terminology is discussed separately in [24].

Case 7: $P \in iS^0, P \notin S^1, i_p = 2k - 1$ and $t_p = 0$ for $k \geq 2$.

The last category of interior points is comprised of the fold boundary points, which are either on the envelopes that are interior to the sweep (see Figure 3), or on the boundary of the initial or final configurations of S . For those fold boundary points that are on one of the *internal envelopes*, the inverted trajectories will not only intersect S^0 , but also be tangent to it (possibly more than once) as shown in Figure 5(h),(j) and (k). Moreover, those that are on the boundary of the initial and final configurations of S will have multiple intersections and tangencies according to definitions 3 and 4 (see Figure 5(g) and (i)).

Definition 7 Given a solid d -dimensional set S and a motion M in \mathbb{E}^d such that $S^0 \cap S^1 = \emptyset$, a point $P \in \mathbb{E}^d$ is a **fold boundary point** of the set sweep(S, M) **iff** one of the following conditions hold:

Case 1: $P \notin S^0, P \notin S^1, i_p = 2k$ and $t_p = j$ for $k \geq 1, j \geq 1$;

Case 2: $P \notin S^0, P \in \partial S^1, i_p = 2k$ and $t_p = j$ for $k \geq 1, j \geq 1$;

Case 3: $P \notin S^0, P \in iS^1, i_p = 2k - 1$ and $t_p = j$ for $k \geq 1, j \geq 1$;

Case 4: $P \in \partial S^0, P \notin S^1, i_p = 2k$ and $t_p = j$ for $k \geq 1, j \geq 1$;

Case 7: $P \in iS^0, P \notin S^1, i_p = 2k - 1$ and $t_p = j$ for $k \geq 1, j \geq 1$;

Note that for interacting fold regions (i.e., fold regions that overlap with other fold regions) Definition 7 will identify the fold boundary points of each such individual fold region (see Figures 6 and 16 in the appendix) even though some of these fold boundary points would be in the interior of other fold regions. This capability becomes important in those cases when only some of these fold regions have a functional role. Alternatively, the fold boundary points that are interior to other fold regions can be classified as fold points, which will have the effect of uniting the interacting fold regions. This is discussed in Appendix A. In addition, for certain shapes and motions (such as a translating “U” shape shown in Figure 6) the set of fold points is partially bounded by subsets of the sweep boundary itself. In these cases, the corresponding fold boundary points are classified as “on” the boundary of the sweep, which is captured by the next definition.

2.3 The “on” points

As mentioned above, the points that are on the boundary of sweep are either on the envelopes or on the boundary of S^0 and S^1 respectively. The first type of points will have an inverted trajectory that has points of tangency, but no intersections with S^0 , as illustrated in Figure 7. The latter category will contain two types of points: those that are on the boundary of S^1 will have an inverted trajectory whose both end points will be, by definition, on the two end configurations of S ; and those that are on the boundary of S^0 which will not have any other intersection point with S^0 . More precisely

Definition 8 Given a solid d -dimensional set S and a motion M in \mathbb{E}^d such that $S^0 \cap S^1 = \emptyset$, a point $P \in \mathbb{E}^d$ is a **sweep boundary point** of set sweep(S, M) **iff** one of the following conditions holds:

Case 1: $P \notin S^0, P \notin S^1, i_p = 0$ and $t_p = j$, for $j \geq 1$;

Case 2: $P \notin S^0, P \in \partial S^1, i_p = 0$ and $t_p = j$, for $j \geq 1$;

Case 4: $P \in \partial S^0, P \notin S^1, i_p = 0$ and $t_p = j$, for $j \geq 1$;

Observe that the points that belong to the interiors of S^0 and S^1 cannot be classified as “on” points, and therefore the corresponding cases do not appear in Definition 8.

2.4 The “out” points

The definition of the “out” points is straightforward: if the intersection between the inverted trajectory of a given point P and S^0 is empty, then there are no points of S that will pass through P during M (see Figure 8). Thus

Definition 9 Given a solid d -dimensional set S and a motion M in \mathbb{E}^d , a point $P \in \mathbb{E}^d$ is an *exterior point* of sweep, or, in other words, belongs to the complement of set sweep(S, M), **iff**

$$\hat{T}_P \cap S^0 = \emptyset.$$

2.5 PMC for sweeping solids

Our decomposition of space into “in,” “on,” and “out” points described above is not only disjoint, but also complete (the union of these sets is the universal set \mathcal{W}). Therefore, we can now state a PMC test for sweeps of moving d -dimensional solids of arbitrary complexity moving according to a one-parameter affine motion M in \mathbb{E}^d :

$$PMC(P, \text{sweep}(S, M)) = \begin{cases} in, & \text{if } P \text{ is a regular, fold,} \\ & \text{or fold boundary point} \\ on, & \text{if } P \text{ is a sweep boundary point} \\ out, & \text{if } \hat{T}_P \cap S^0 = \emptyset \end{cases} \quad (7)$$

3 Completeness of our Point Membership Classification

The completeness of our classification can be conventionally proved based on set theoretic principles. However, a more intuitive proof can be developed graphically as illustrated in Figure 9. First, recall that the number of intersections i_p and the number of tangencies t_p are both natural numbers and therefore the axes shown in Figure 9 contain discrete values.

According to our classification, a given point of the space can be a

- regular (definition 5) and fold (definition 6) point;
- a fold-boundary (definition 7) point;
- a sweep boundary point (definition 8), or

- an exterior point of the sweep (definition 9).

Figure 9 contains all possible (i.e. permissible) combinations of i_p and t_p according to this classification. More importantly, this figure shows that: (1) there are no combinations of i_p and t_p that do not belong to at least one classification, which implies that our classification is complete, and (2) the same combination of i_p and t_p can correspond to only one category.

The practical implications of this completeness becomes apparent if we recall that a point on the actual sweep boundary will have an inverted trajectory that will be tangent (possibly more than once) to the moving object in its initial configuration. It is known that computing these tangent points not only can be more time consuming than computing intersections, but is also less robust. On the other hand, the completeness of our classification ensures that one can use Definitions 5-9 to classify a point of the space as an “on” point if it is neither “in” nor “out” relative to the sweep up to some small ε . Specific implementation issues are beyond the scope of this paper.

4 Examples

In this section we describe several examples that verify our prototype implementation and illustrate some of the main capabilities of our PMC test. Note that our classification is generic and is not limited by the class of shapes and motions illustrated in this section. We implemented the PMC test described above in the Parasolid Workshop prototyping tool by Siemens PLM software using Microsoft Visual Studio.

It is important to note that, since our classification is independent of specific implementations of curve-solid intersections, the output of specific implementations for the cases when the intersection is not “clean” (see the intersections shown in Figure 2) may be different from what our classification requires. A typical situation is the case when the solid is given as a boundary representation, and the curve-solid intersection reduces to a series of curve-face intersections with all the faces of the solid. For example, if the curve is tangent to a cube at a vertex, such that $i_p = 0$ and $t_p = 1$ (see Figure 2), the curve may appear to intersect all three faces of the cube since the vertex is part of three separate faces, but the correct number of intersections and tangencies for such a curve should be zero and one, respectively.

We verified our implementation separately for 2 and 3 dimensional sweeps due to the lack of algorithms for generating the boundary of 3D sweeps in current commercial CAD software. For the planar case (2D solids moving in the same plane), we compared our results with the boundary of the 2D sweep as approximated by a commercial CAD package. Since the 3D sweep boundary evaluation is not available in any commercial CAD system, we compared the computed sweep boundary points with the boundary of discrete instances of solid S along the motion as shown in Figures 10-15.

The first example in Figure 10 shows a spur gear moving relative to its meshing gear (not shown) in the plane. The gear ratio for this example is 1.5 and the gear shown has 20 teeth. The relative motion between the gears is a rigid body motion that contains a translation along a circular trajectory T , and a uniform rotation around its centroid by 5π . We show in Figure 10 the inverted trajectories of only 3 test points to avoid the cluttering of the illustration. It can be seen that the inverted trajectory of an *interior point* P_1 intersects S^0 at least 4 times, so P_1 is a fold point. However, the inverted trajectory of an exterior point P_3 does not intersect the gear in its original configuration. On the other hand, the inverted trajectory for point P_2 has one point of tangency

(or more depending on the number of complete revolutions of the displayed gear relative to its meshing counterpart) with the original set.

Our second example in Figure 11 shows a sphere moving in 3D space, translating along a prescribed trajectory as well as rotating by π around the z axis. This 3-dimensional sweep has been computed by sweeping a great circle of the sphere through intermediate sections that are perpendicular to the shown trajectory, which is supported by typical high-end commercial CAD systems. The approximated boundary of this 3D sweep self-intersects itself in two separate regions, which indicates that the corresponding envelopes have singularities, and that the sphere loses contact (locally) with its envelopes. Consequently, fold and fold boundary points are expected to occur in these regions. All three different interior points as well as “on” and “out” points have been classified based on the number of intersections and tangencies between their inverted trajectories and the original sphere.

In the third example shown in Figure 12, the centroid of disk S translates along trajectory T with no scaling (a) or with uniform scaling according to some analytic law (b). This figure shows the computed boundary of the set swept by S , which corresponds to offset curves of T . For the case shown in Figure 12(a) these offset curves self-intersect and generate fold regions, but the scaling illustrated in Figure 12(b) eliminates the self-intersections of the envelope curves.

The fourth example shown in Figure 13 illustrates a 3-dimensional free-form solid moving in 3D space according to an affine motion that also includes scaling. The color coded intermediate configurations of the object are shown in Figure 13(a) and they correspond to the color coding of the point cloud illustrated in Figure 13(b).

Our next example illustrated in Figure 14 shows the insertion of a toner cartridge into a printer, whose motion can be easily obtained by interpolating a set of discrete configurations. Several color-coded intermediate configurations of the cartridge as well as the point cloud corresponding to the boundary of the set swept by the cartridge during the insertion are displayed in Figure 14(b) and (c). This point cloud can, for example, be used to compute potential collisions and interference during the insertion. Note that such a test does not require object tessellation or any other specific geometric representation as long as the geometric representation supports curve-surface intersection.

The last example of Figure 15(a) shows a custom-made cutter (bounded by NURBS surfaces) as it moves during a 5-axis machining simulation to compute the machined volume for a slot-milling operation. The cutting tool is modeled as a (NURBS) surface of revolution, and the tip of the cutter moves along a prescribed trajectory T while rotating around x and y axes by $\pi/8$ and $\pi/6$ respectively. We superimposed several color-coded intermediate configurations of the cutter as seen in Figure 15(b). The front, side and isometric views of the computed point cloud for the sweep boundary can be seen in Figure 15(c), which can be used to compute the machined volume or perform validation of the corresponding path planning algorithms.

Importantly, the “undercut” or “overcut” regions, as well as local and global self intersections of the swept set can be detected and quantified from the information extracted from the proposed curve-solid intersection as discussed in [24]. This classification has important practical applications in manufacturing, for example in on-line path planning, tool path generation and machining verification for both 3- and 5-axis CNC machining, calculation of offset curves, as well as in mechanical design - for example in the design of higher pairs.

5 Conclusions

Sweeping a moving solid through space has found many applications not only as an effective design tool for creating highly complex three-dimensional shapes, but also in computer aided manufacturing, as well as in robotics and computer graphics to name just a few. However, computing the boundary of the set swept by a moving solid remains, in general, difficult, and computational support for solid sweeping remains vastly inadequate. This deficiency is primarily due to an incomplete characterization of the validity and computational properties of sweeps, or equivalently, to the lack of a generic point membership test for solid sweeping.

In this paper we introduce a generic Point Membership Classification procedure for sweeping both 2- and 3-dimensional solids according to a motion in a space having the same dimension. Our approach relies on performing an intersection between the inverted trajectory of a point in space and the moving object in its original *static* configuration followed by a careful postprocessing of the intersection results. This procedure allows the classification of any point of space as being in the *interior* (i.e., regular, fold or fold boundary point), on the *boundary* or in the *complement* of the set swept by the moving solid, and can be applied to arbitrarily complex solids moving according to both rigid and non-rigid motions.

From a computational point of view, this PMC procedure can be implemented in any geometric representation that supports curve-solid intersections. To a great extent, we avoided making any assumptions on the moving object and its motion through space, which implies that our results remain widely applicable. This also means that a number of specific issues have not yet been addressed. For example, specific computational properties, such as computational cost and robustness, will depend on and vary between individual geometric and motion representations. However, the correctness of our mathematical model is independent of specific computational choices or properties.

It is important to note that this classification can be used either as a stand-alone boundary evaluator for sweeps together with an appropriate sampling algorithm (such as the octree-based sampling that we discuss in [28]), or in conjunction with other boundary evaluation algorithms. Note that our classification leads to a finer decomposition than the usual PMC tests for geometric shapes and has important applications in engineering design and manufacturing. More generally, our classification establishes the foundations for developing a new generation of computational tools for sweep boundary evaluation and trimming, as well as a number of practical applications such as shape synthesis, contact analysis [29] and path planning.

Acknowledgments

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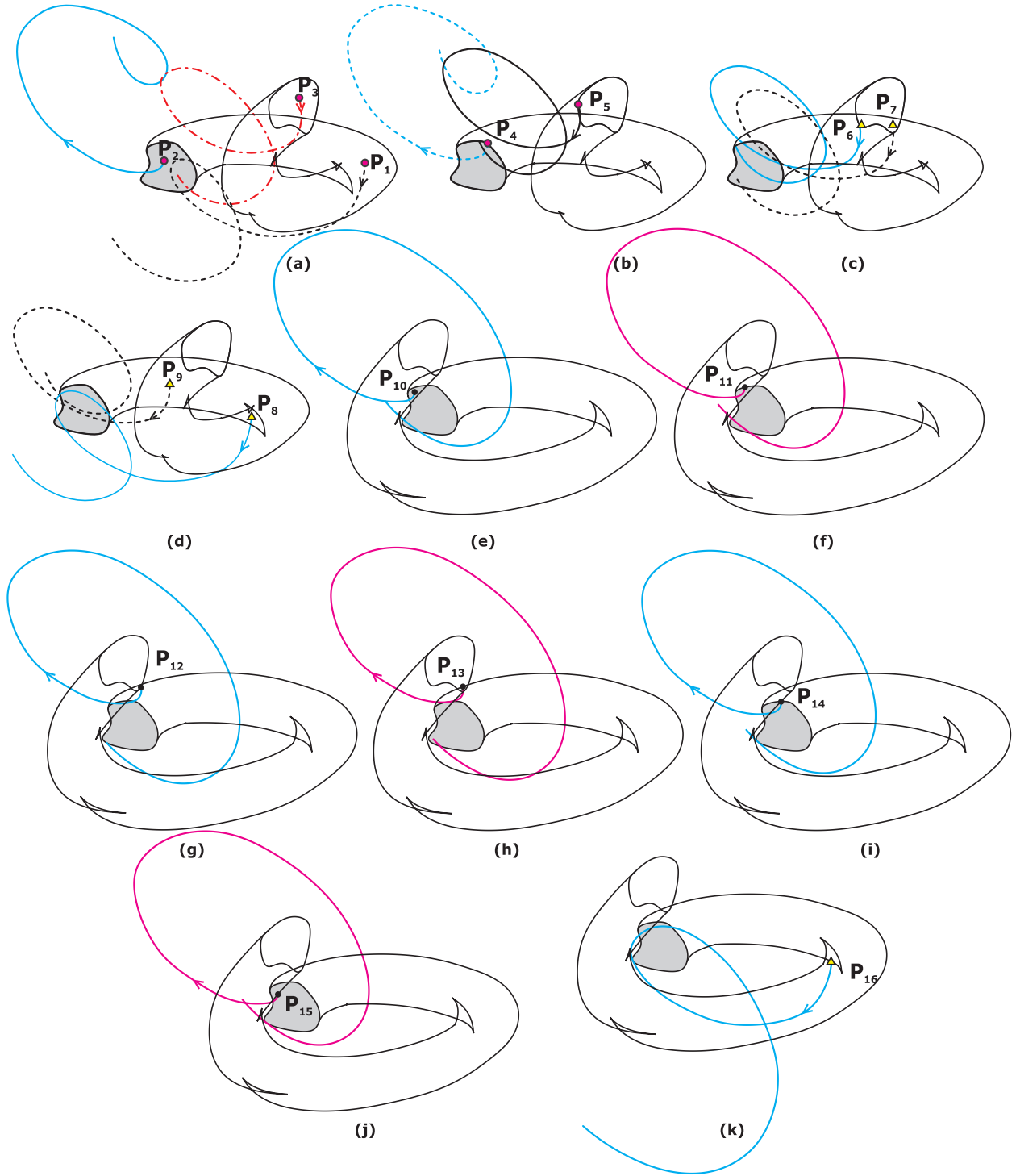


Figure 5: The interior points of sweep as well as their inverted trajectories: figures (a), (b) show regular points; (c), (d), (e) and (f) fold points. The illustrated points correspond to cases from Definition 5 as follows: $(P_1, 1)$, $(P_2, 7)$, $(P_3, 3)$, $(P_4, 4)$, and $(P_5, 2)$; from Definition 6 as follows: $(P_6, 2)$, $(P_7, 3)$, $(P_8, 1)$, $(P_9, 1)$, $(P_{10}, 7)$, and $(P_{11}, 4)$. Figures (g), (h), (i), (j), and (k) illustrate fold boundary points corresponding to cases from Definition 7 as follows: $(P_{12}, 2)$, $(P_{13}, 3)$, $(P_{14}, 4)$, $(P_{15}, 7)$, and $(P_{16}, 1)$.

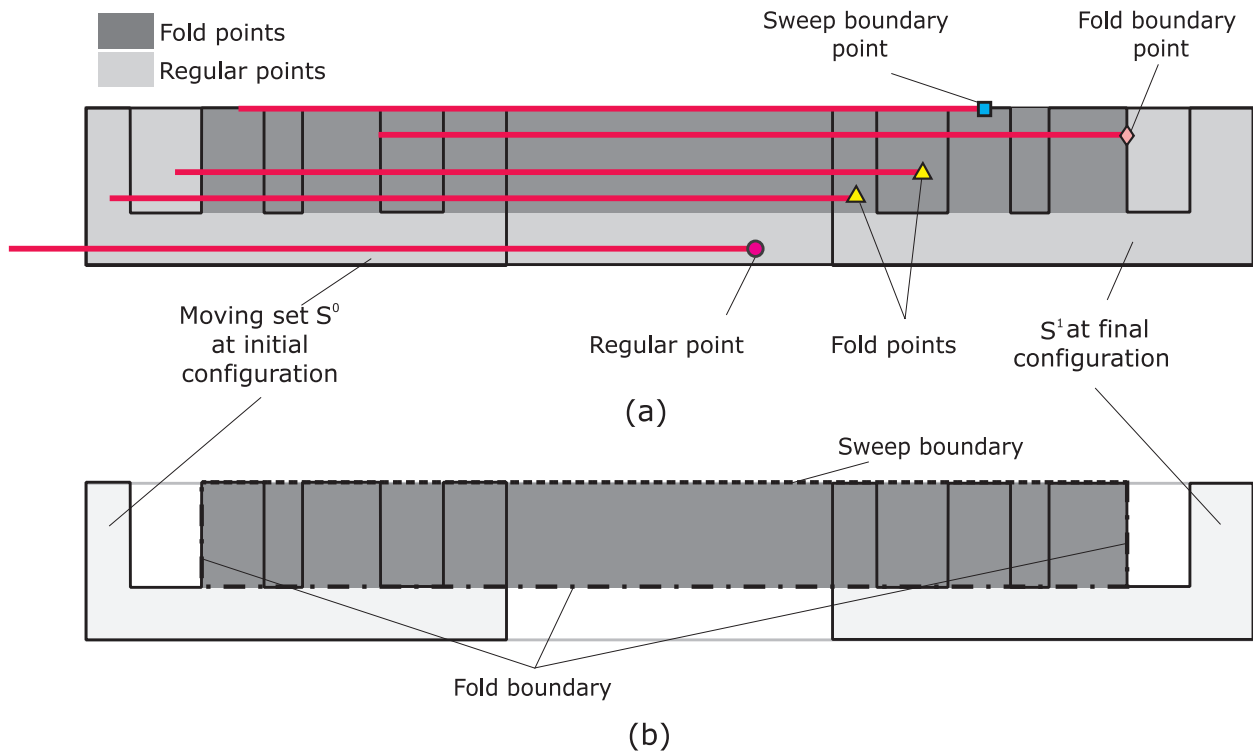


Figure 6: Subsets of the boundary of the fold region can be on the boundary of the sweep itself. These points are classified as “on” points.

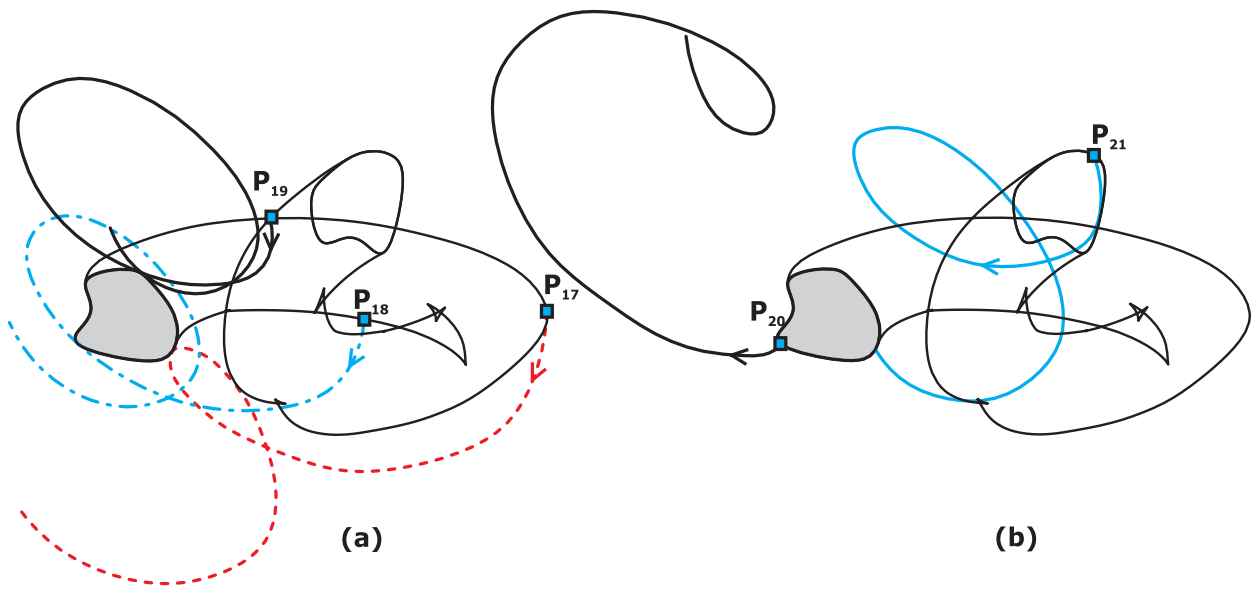


Figure 7: Boundary points of sweep as well as their inverted trajectories: figures (a) and (b) show several “on” points that correspond to cases from Definition 8 as follows: $(P_{17}, 1)$, $(P_{18}, 1)$, $(P_{19}, 1)$, $(P_{20}, 4)$, and $(P_{21}, 2)$.

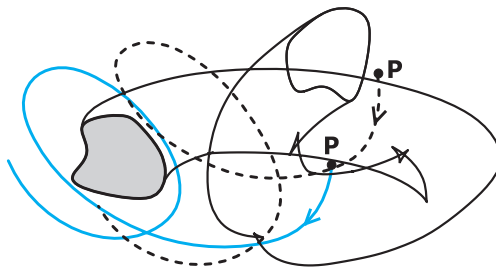
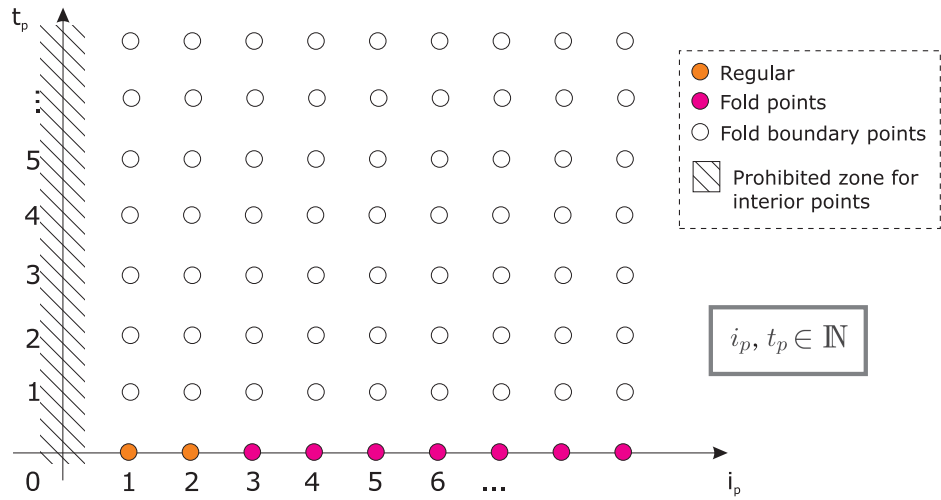
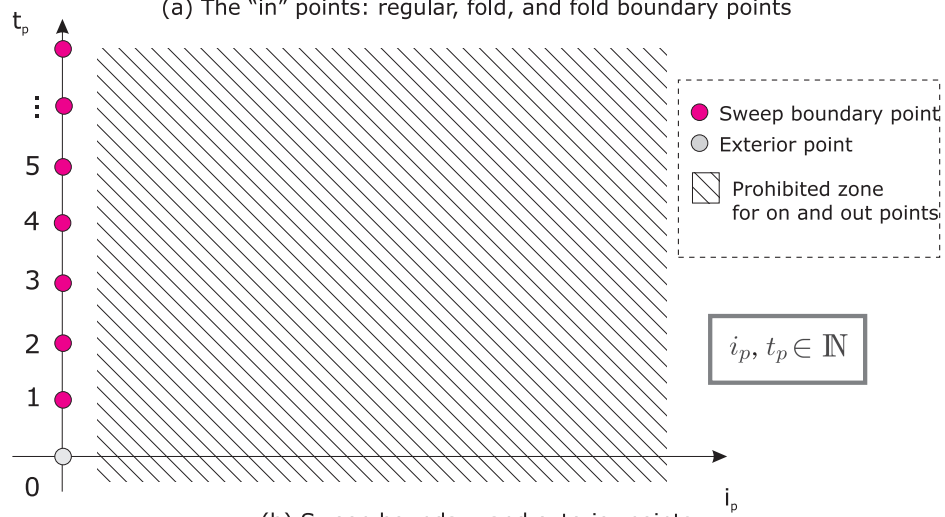


Figure 8: Examples of points that are classified as “out” relative to the sweep.



(a) The "in" points: regular, fold, and fold boundary points



(b) Sweep boundary and exterior points

Figure 9: Completeness of the point membership classification. There are no combinations of i_p and t_p that do not belong to at least one classification.

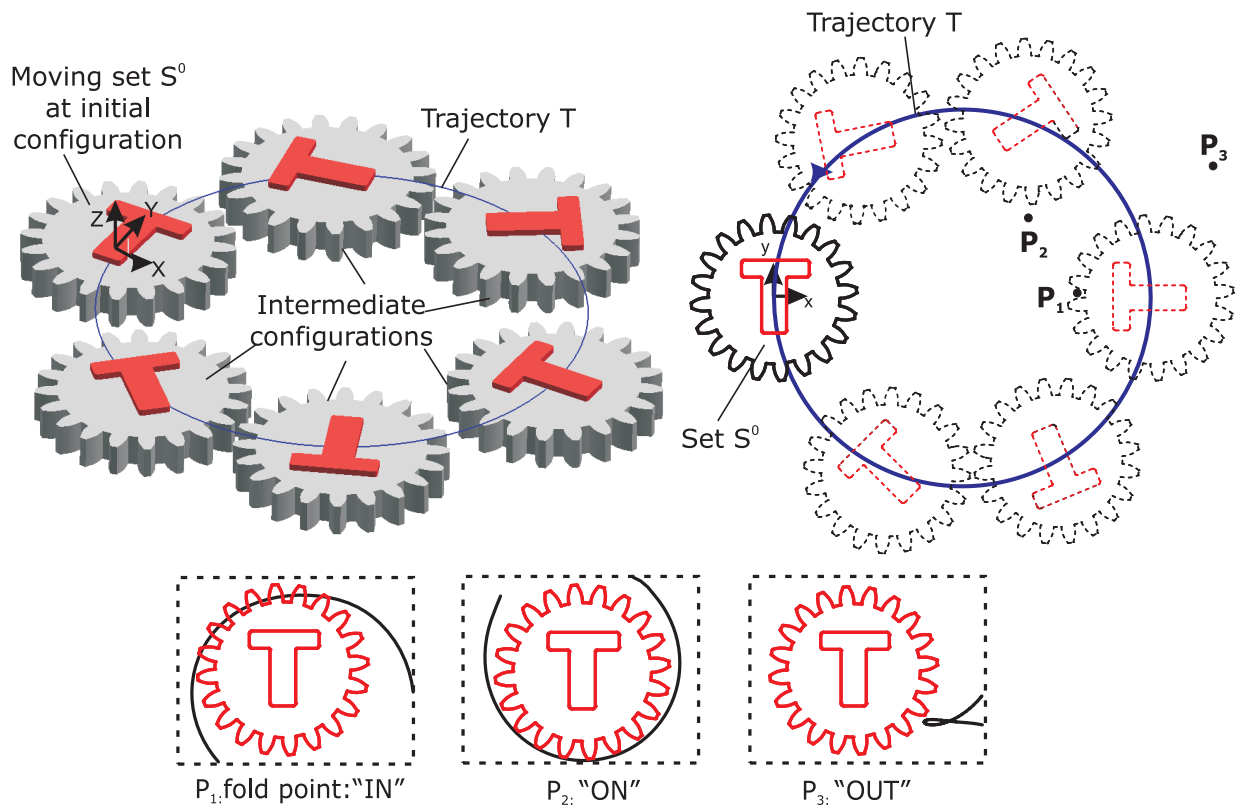


Figure 10: A spur gear and its relative motion.

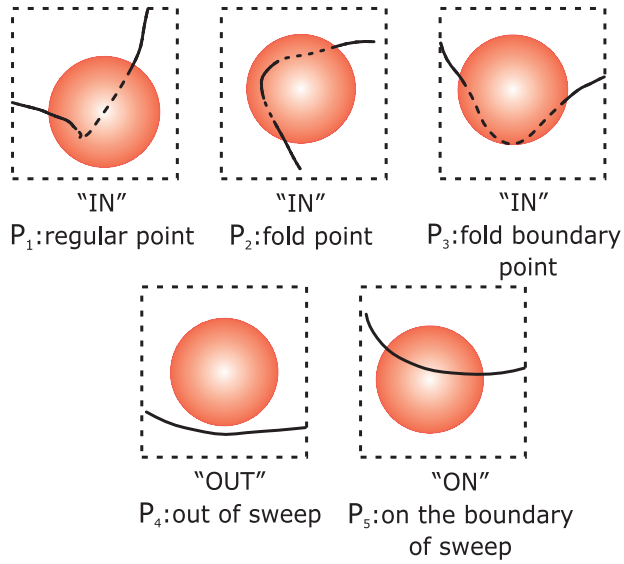
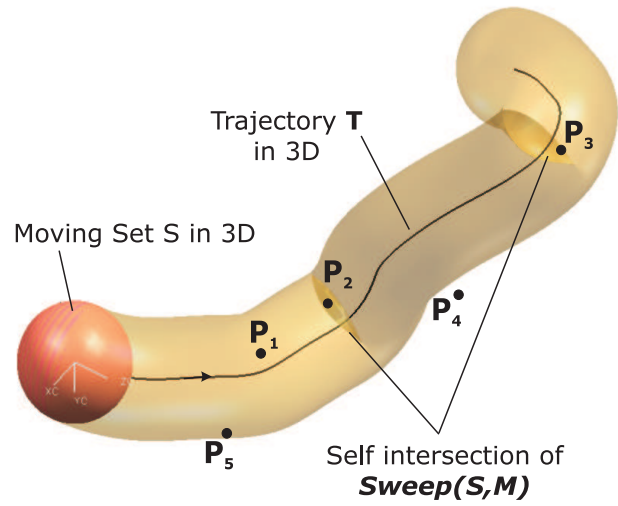


Figure 11: A sphere moving in the 3D space according to a rigid body motion

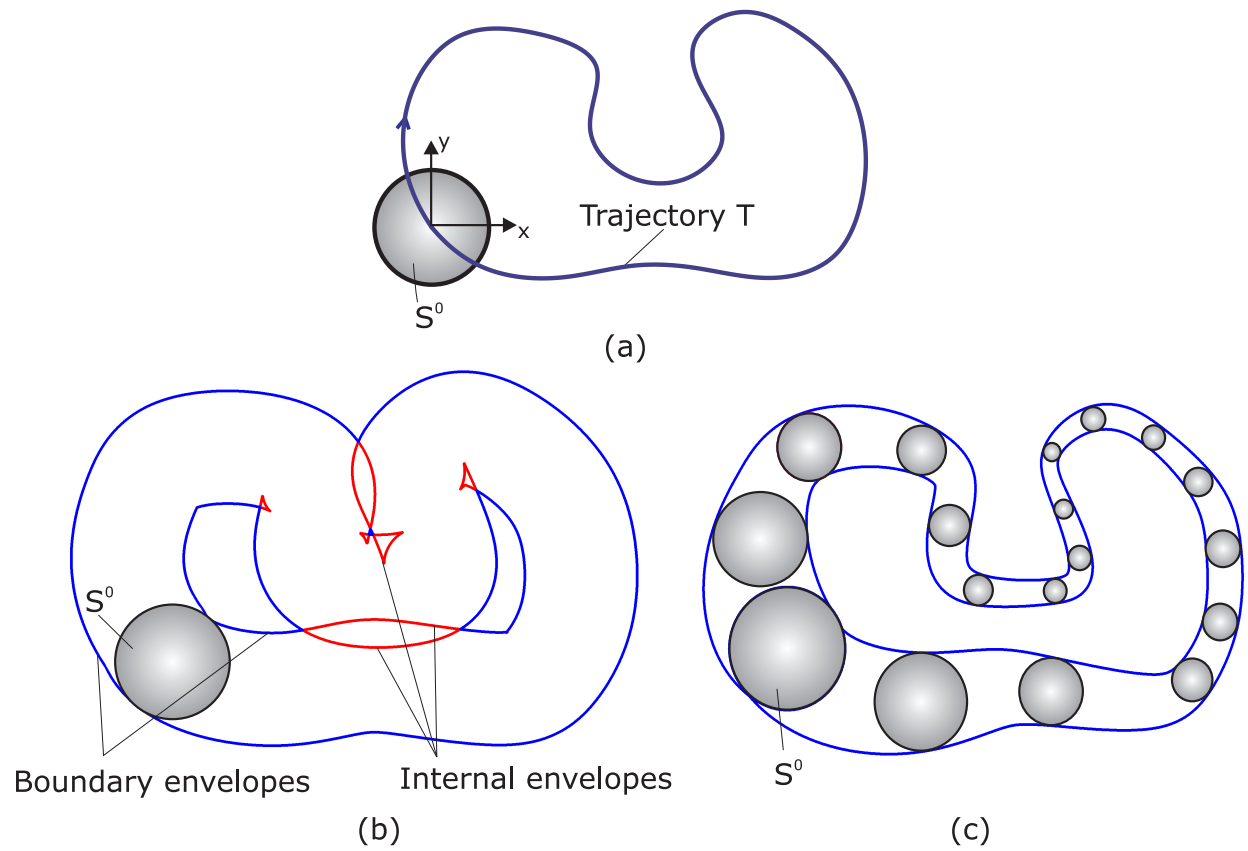


Figure 12: Computing offset curves with and without self-intersections.

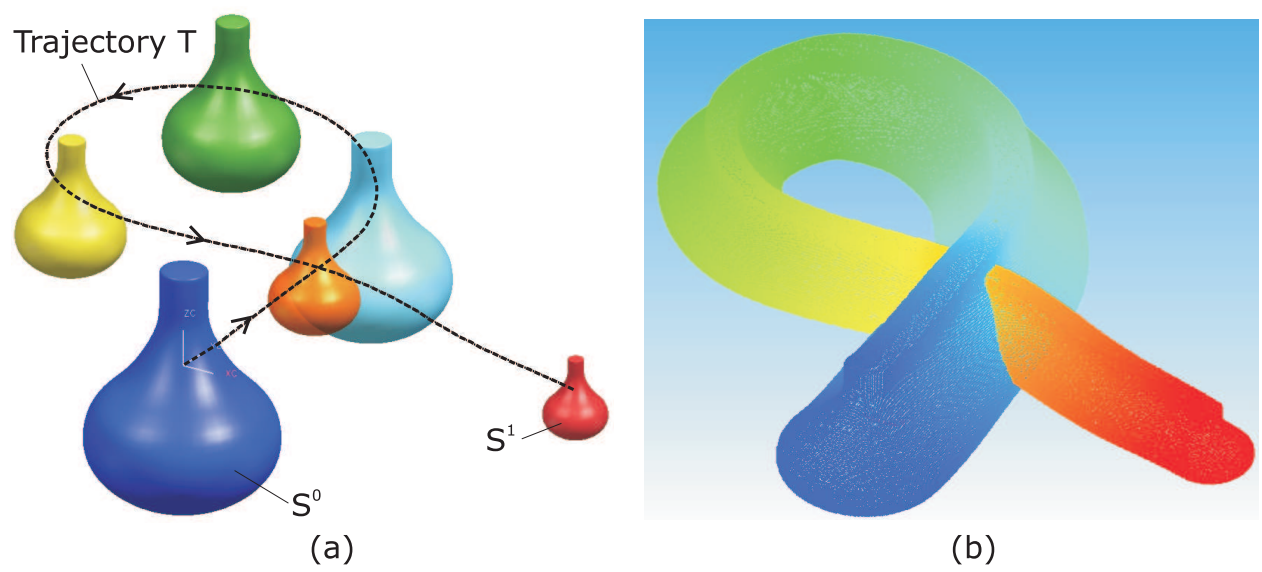


Figure 13: A free-form object deforms as it moves and the point cloud corresponding to the boundary of the set swept by the moving object. Observe that the 3D swept set self-intersects itself.

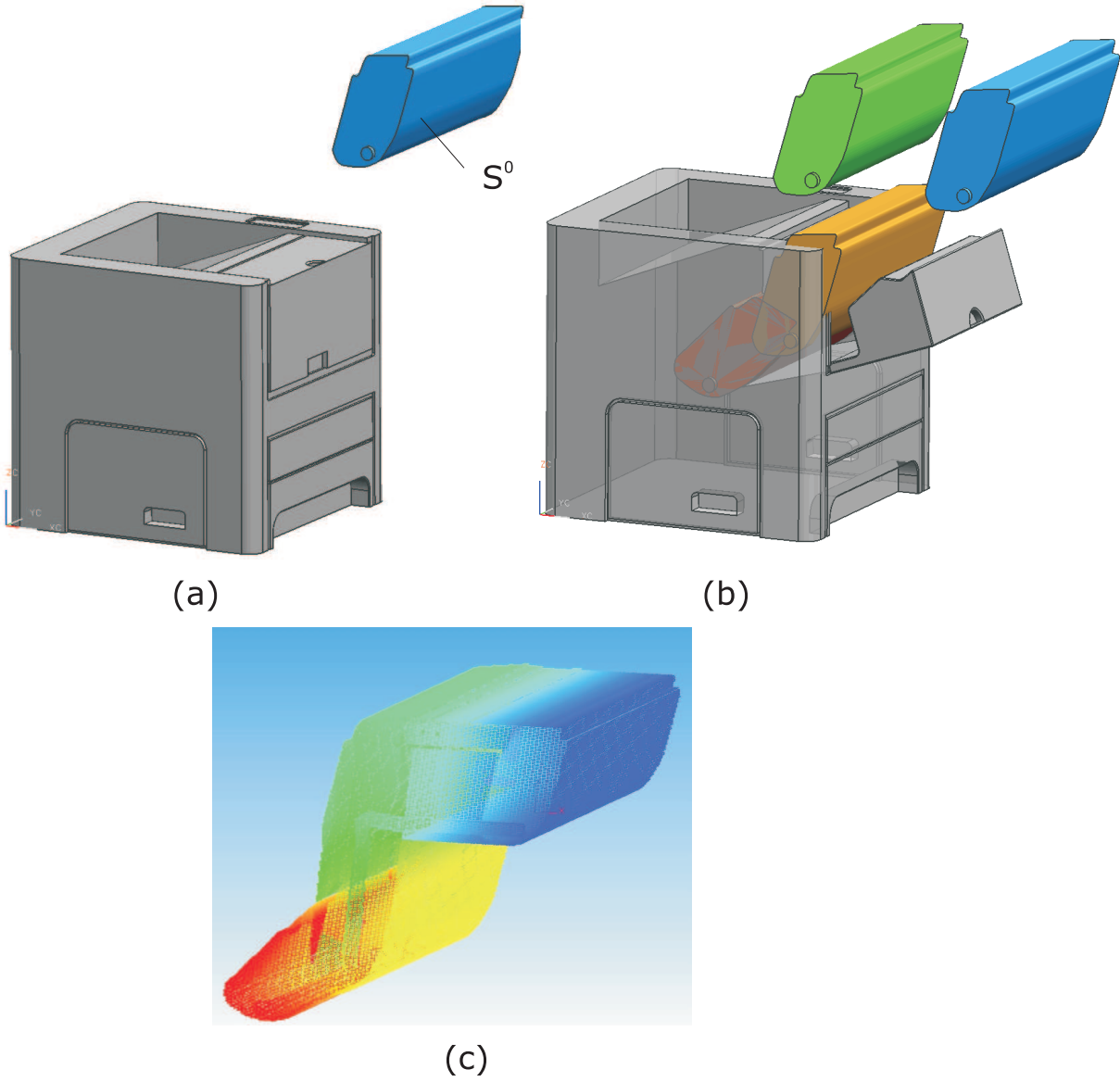


Figure 14: A toner cartridge as it is being inserted into the printer, and its sweep boundary points

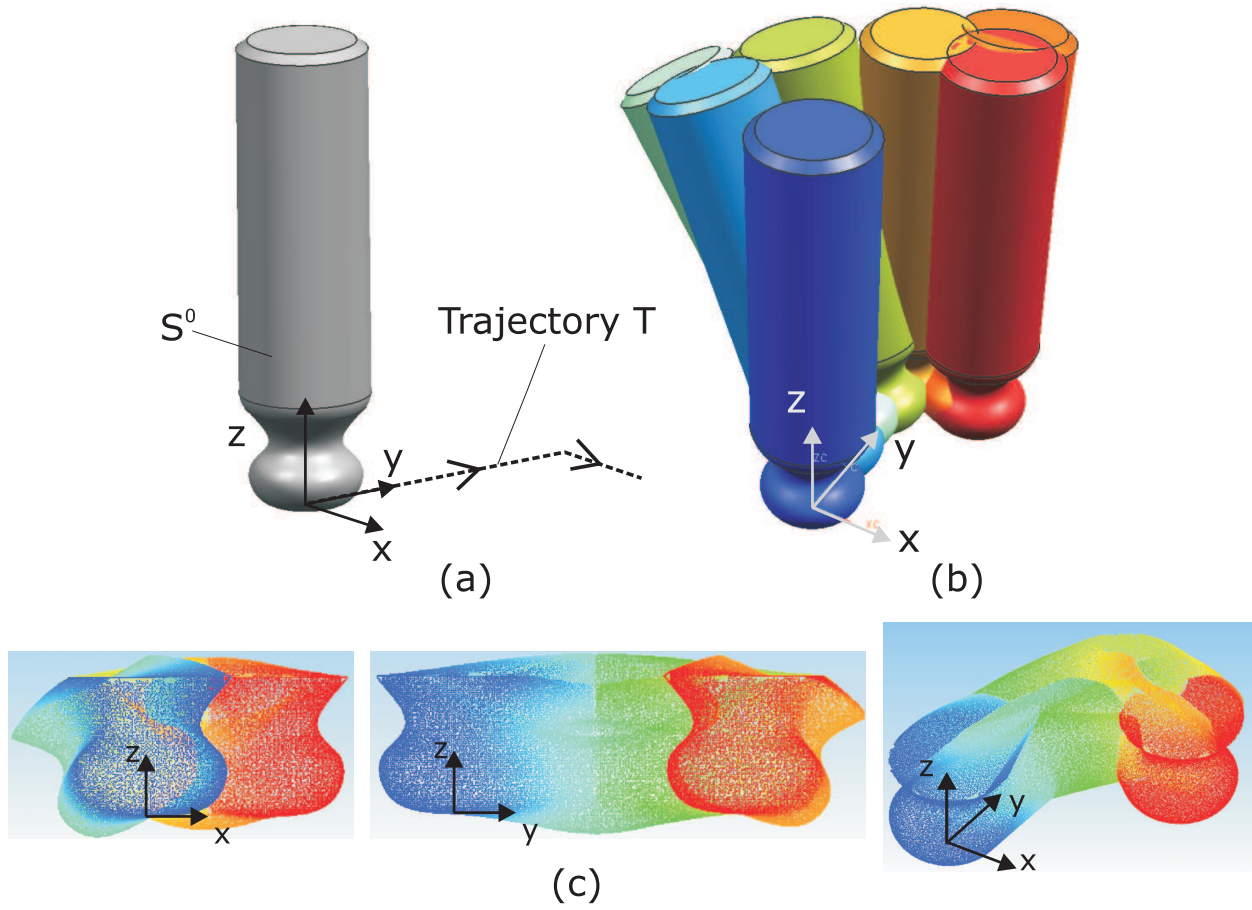


Figure 15: A milling cutter moving through space during a 5-axis CNC simulation. Two side views and an isometric view of the computer sweep boundary points are shown in (c).

A Appendix: Combining individual fold regions

As mentioned in Section 2, certain shapes and motions will generate interacting fold regions (i.e., fold regions that overlap with other fold regions), and Definition 7 identifies the fold boundary points of each such individual fold region (see Figure 6) even though some of these fold boundary points would be in the interior of other fold regions. These fold boundary points can be classified as fold points, which will have the effect of uniting the interacting fold regions as follows:

Definition 10 Given a solid d -dimensional set S and a motion M in \mathbb{E}^d such that $S^0 \cap S^1 = \emptyset$, a point $P \in \mathbb{E}^d$ is a **fold point** of the set sweep(S, M) **iff** one of the following conditions hold:

Case 1: $P \notin S^0, P \notin S^1, (i_p = 2k \text{ and } t_p = 0)$ or $(i_p = 2k \text{ and } t_p = j)$ for $k \geq 2, j \geq 1$;

Case 2: $P \notin S^0, P \in \partial S^1, (i_p = 2k \text{ and } t_p = 0)$ or $(i_p = 2k \text{ and } t_p = j)$ for $k \geq 2, j \geq 1$;

Case 3: $P \notin S^0, P \in iS^1, (i_p = 2k - 1 \text{ and } t_p = 0)$ or $(i_p = 2k - 1 \text{ and } t_p = j)$ for $k \geq 2, j \geq 1$;

Case 4: $P \in \partial S^0, P \notin S^1, (i_p = 2k \text{ and } t_p = 0)$ or $(i_p = 2k \text{ and } t_p = j)$ for $k \geq 2, j \geq 1$;

Case 7: $P \in iS^0, P \notin S^1, (i_p = 2k - 1 \text{ and } t_p = 0)$ or $(i_p = 2k - 1 \text{ and } t_p = j)$ for $k \geq 2, j \geq 1$;

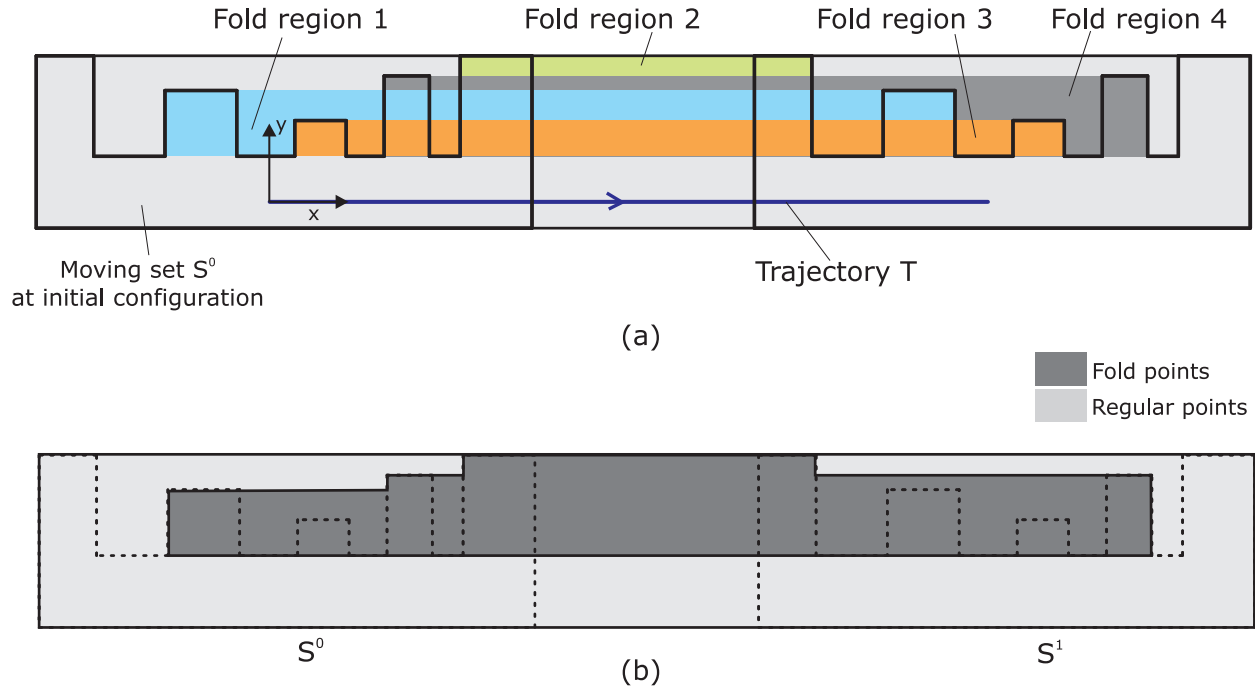


Figure 16: Point membership classification for interacting fold regions: (a) individual fold regions can overlap and fold boundary points are determined according to definitions 6 and 7; (b) The individual fold regions are combined according to definitions 10 and 11.

Definition 11 Given a solid d -dimensional set S and a motion M in \mathbb{E}^d such that $S^0 \cap S^1 = \emptyset$, a point $P \in \mathbb{E}^d$ is a **fold boundary point** of the set $\text{sweep}(S, M)$ **iff** one of the following conditions hold:

Case 1: $P \notin S^0, P \notin S^1, i_p = 2k$ and $t_p = j$ for $k = 1, j \geq 1$;

Case 2: $P \notin S^0, P \in \partial S^1, i_p = 2k$ and $t_p = j$ for $k = 1, j \geq 1$;

Case 3: $P \notin S^0, P \in iS^1, i_p = 2k - 1$ and $t_p = j$ for $k = 1, j \geq 1$;

Case 4: $P \in \partial S^0, P \notin S^1, i_p = 2k$ and $t_p = j$ for $k = 1, j \geq 1$;

Case 7: $P \in iS^0, P \notin S^1, i_p = 2k - 1$ and $t_p = j$ for $k = 1, j \geq 1$;

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